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Local and Global Approximation Theorems for Positive Linear Operators

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In this paper we bridge local and global approximation theorems for positive linear operators via Ditzian–Totik moduli $\omega_\phi^2(f, \delta)$ of second order whereby the step-weights ϕ are functions whose squares are concave. Both direct and converse theorems are derived. In particular we investigate the situation for exponential-type and Bernstein-type operators. © 1998 Academic Press

1. INTRODUCTION

In [6] it was shown that for the Bernstein operator $(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x)\, f(k/n)$, $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, the estimate

$$|f(x) - (B_n f)(x)| \leqslant C\omega_{\varphi^\lambda}^2(f, n^{-1/2}\varphi(x)^{1-\lambda}), \qquad x \in I = [0, 1], \qquad (1)$$

holds true, where $\lambda \in [0, 1]$, $\varphi(x) = \sqrt{x(1-x)}$, and the Ditzian–Totik modulus of smoothness of second order is given by

$$\omega_\phi^2(f, \delta) := \sup_{|h| \leqslant \delta} \sup_{x \pm h\phi(x) \in I} |f(x - \phi(x)\, h) - 2f(x) + f(x + \phi(x)\, h))|,$$

in which ϕ: $[0, 1] \to R$ is an admissible step-weight function (for details see [9]).

The case $\lambda = 0$ in (1) gives the classical local estimate whereas $\lambda = 1$ gives the global norm estimate developed by Ditzian and Totik. Therefore (1) bridges the gap between the local and global approximation theorems for the Bernstein operator. Such results for polynomial approximation were previously investigated by [7, 8 and 14].

Inequality (1) shows that the error $f(x) - (B_n f)(x)$ is bounded pointwise by $C(n^{-1/2}\varphi(x)^{1-\lambda})^\alpha$ if $\omega_{\varphi^\lambda}^2(f, \delta) = O(\delta^\alpha)$ and $\alpha \in [0, 2]$. It can be seen from [5, 16] that the converse result also holds true. With this,

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$\omega^2_{\varphi^\lambda}(f, \delta) = O(\delta^\alpha)$ can be characterized in terms of the Bernstein operator; that is, the equivalence

$$|f(x) - (B_n f)(x)| = O((n^{-1/2}\varphi(x)^{1-\lambda})^\alpha) \Leftrightarrow \omega^2_{\varphi^\lambda}(f, \delta) = O(\delta^\alpha)$$

holds true for all $\alpha \in (0, 2)$ and $\lambda \in [0, 1]$.

Moreover, in [10] inequality (1) could be further extended to the more general inequality

$$|f(x) - (B_n f)(x)| \leqslant C\omega^2_\varphi\left(f, n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right), \qquad x \in [0, 1], \tag{2}$$

whereby ϕ: $[0, 1] \to R$ is an admissible step-weight function of the Ditzian–Totik modulus (see [9]) and ϕ^2 is a concave function. Obviously (2) includes (1) if ϕ is replaced by φ^λ, $\lambda \in [0, 1]$. It was also proved in [10] that an inverse result to (2) holds true; i.e., $\alpha \in (0, 2)$ and

$$|f(x) - (B_n f)(x)| \leqslant C_1\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^\alpha, \qquad x \in [0, 1], \quad n = 1, 2, ...,$$

implies $\omega^2_\phi(f, \delta) \leqslant C_2\delta^\alpha$ if, in addition, φ^2/ϕ^2 is concave, which is fulfilled for $\phi = \varphi^\lambda$, $\lambda \in [0, 1]$, in particular.

In this paper we investigate the situation more generally for other positive linear operators. Section 2 provides direct estimates for arbitrary positive linear operators which will be applied to exponential-type and Bernstein-type operators. In particular we obtain similar estimates (2) for the Szász-Mirakjan, Kantorovich-type and Durrmeyer-type operators. We also show that similar estimates hold true for a class of Bernstein-type operators which are "not far away" from the original Bernstein operator. Section 3 establishes inverse results for positive linear operators.

2. DIRECT RESULTS

We shall study here positive linear operators of continuous functions which are defined on finite or infinite intervals $I \subset R$. Without loss of generality, let I be one of the intervals $[0, 1]$, R_0^+ or R. Different intervals can be obtained by affine linear substitutions.

We will use the weighted K-functional of second order for $f \in C(I)$ defined by

$$K^2_\phi(f, \delta^2) := \inf_{g' \in AC_{loc}(I)} (\|f - g\| + \delta^2\,\|\phi^2 g''\|), \qquad \delta \geqslant 0,$$

in which $\|\cdot\|$ denotes the uniform norm on I and $g' \in AC_{loc}(I)$ means that g is differentiable and g' is absolutely continuous in every closed finite interval $[a, b] \subset I$. Moreover, the Ditzian–Totik moduli of first order are given by

$$\omega_\phi(f, \delta) := \sup_{|h| \leqslant \delta} \sup_{x \pm (h/2)\, \phi(x) \in I} \left| f\left(x + \phi(x)\frac{h}{2}\right) - f\left(x - \phi(x)\frac{h}{2}\right)\right|$$

and

$$\vec{\omega}_\phi(f, \delta) := \sup_{|h| \leqslant \delta} \sup_{x,\, x + h\phi(x) \in I} |f(x + \phi(x)\, h) - f(x)|.$$

Given that $\phi: I \to R$ is an admissible step-weight function it is well known that K-functional $K^2_\phi(f, \delta^2)$ and Ditzian–Totik modulus $\omega^2_\phi(f, \delta)$ are equivalent. Likewise $\vec{\omega}_\phi(f, \delta)$ and $\omega_\phi(f, \delta)$ are equivalent (cf. [9]).

Now we can state the following result.

THEOREM 1. *Let $I \in \{[0, 1], R_0^+, R\}$ and $A: C(I) \to C(I)$ be a bounded positive linear operator which preserves constants. In case $I = [0, 1]$ let $|A(\bullet - x)(x)| \leqslant \frac{1}{2}$. If $\psi, \phi: I \to R$ are functions with ψ an admissible step-weight function of the Ditzian–Totik modulus and with ϕ^2 concave then the pointwise approximation*

$$|f(x) - (Af)(x)| \leqslant \vec{w}_\psi\left(f, \frac{|A(\bullet - x)(x)|}{\psi(x)}\right) + 4K^2_\phi\left(f, \frac{A((\bullet - x)^2)(x)}{\phi^2(x)}\right)$$

holds true for $x \in I$ and $f \in C(I)$.

Proof. First we construct a new operator $\tilde{A} = A + L$ which also preserves linear functions. For that let $(\vec{\Delta}_u f)(x) = f(x+u) - f(x)$ and

$$(Lf)(x) = \begin{cases} -\vec{\Delta}_{A(\bullet - x)(x)} f)(x), & \text{if } \; x + A(\bullet - x)(x) \in I \\ (\vec{\Delta}_{-A(\bullet - x)(x)} f)(x), & \text{if } \; x + A(\bullet - x)(x) \notin I \end{cases} \tag{3}$$

for $x \in I$. (Because of assumption $|A(\bullet - x)(x)| \leqslant \frac{1}{2}$ for $I = [0, 1]$ the right side of (3) is well-defined for all $x \in [0, 1]$.) A simple calculation shows that $\tilde{A}1 = 1$ and $\tilde{A}(\bullet - x)(x) = 0$.

Now let $x \in I$ be fixed and $g: I \to R$ with $g' \in AC_{loc}(I)$ be arbitrary. From Taylor's expansion

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t g''(s)(t - s)\, ds, \qquad t \in I,$$

we see that

$$(\tilde{A}g)(x) - g(x) = \tilde{A}\left(\int_x^{\bullet} g''(s)(\bullet - s)\, ds\right)(x). \tag{4}$$

For each $s = t + \tau(x - t)$, $\tau \in [0, 1]$, we can estimate (using the concavity of ϕ^2)

$$\frac{|t-s|}{\phi^2(s)} = \frac{\tau|x-t|}{\phi^2(t+\tau(x-t))} \leqslant \frac{\tau|x-t|}{\phi^2(t)+\tau(\phi^2(x)-\phi^2(t))} \leqslant \frac{|x-t|}{\phi^2(x)}. \tag{5}$$

Therefore

$$\begin{aligned}\left|\int_x^t g''(s)(t-s)\, ds\right| &\leqslant \|\phi^2 g''\| \left|\int_x^t \frac{|t-s|}{\phi^2(s)}\, ds\right| \\ &\leqslant \|\phi^2 g''\| \left|\int_x^t \frac{|t-x|}{\phi^2(x)}\, ds\right| \\ &= \|\phi^2 g''\| \frac{(t-x)^2}{\phi^2(x)} \end{aligned} \tag{6}$$

and by positivity of A

$$A\left(\int_x^{\bullet} g''(s)(\bullet - s)\, ds\right)(x) \leqslant \|\phi^2 g''\| \frac{A((\bullet - x)^2\,(x)}{\phi^2(x)}.$$

Although the operator L is not positive we can estimate by means of (6) as

$$\begin{aligned}\left|\vec{\Delta}_u\left(\int_x^{\bullet} g''(s)(\bullet - s)\, ds\right)(x)\right| &= \left|\int_x^{x+u} g''(s)(x+u-s)\, ds\right| \\ &\leqslant \|\phi^2 g''\| \frac{u^2}{\phi^2(x)}\end{aligned}$$

and therefore with $u := \pm A(\bullet - x)(x)$

$$\left|L\left(\int_x^{\bullet} g''(s)(\bullet - s)\, ds\right)(x)\right| \leqslant \|\phi^2 g''\| \frac{(A(\bullet - x))^2\,(x)}{\phi^2(x)}.$$

Moreover, by application of the Cauchy–Schwarz inequality we have $|(Af)(x)| \leqslant \sqrt{(A)(f^2)(x)}$ (since for the positive operator A we have the representation $(Af)(t) = \int_I f(u)\, d\alpha_t(u)$, where α_t is increasing and satisfying $\int_I d\alpha_t(u) = 1$). This gives

$$(A(\bullet - x))^2\,(x) \leqslant A((\bullet - x)^2)(x). \tag{7}$$

Together with (4) we arrive at

$$|(\tilde{A}g)(x) - g(x)| \leqslant \frac{2A((\bullet - x)^2)(x)}{\phi^2(x)}\, \|\phi^2 g''\|$$

for all g with $g' \in AC_{loc}(I)$.

Because $\|\tilde{A}\| \leqslant \|A\| + \|L\| \leqslant 3$ we obtain, for $f \in C(I)$,

$$\begin{aligned} |(\tilde{A}f)(x) - f(x)| &\leqslant |(\tilde{A}(f-g))(x)| + |g(x) - f(x)| + |(\tilde{A}g)(x) - g(x)| \\ &\leqslant 4\left(\|f - g\| + \frac{A((\bullet - x)^2)(x)}{\phi^2(x)}\, \|\phi^2 g''\|\right). \end{aligned}$$

Taking the infimum on the right hand side over all g with $g' \in AC_{loc}(I)$ we obtain

$$|(\tilde{A}f)(x) - f(x)| \leqslant 4K_\phi^2\left(f, \frac{A((\bullet - x)^2)(x)}{\phi^2(x)}\right)$$

and consequently,

$$|(Af)(x) - f(x)| \leqslant |(Lf)(x)| + 4K_\phi^2\left(f, \frac{A((\bullet - x)^2)(x)}{\phi^2(x)}\right). \tag{8}$$

Because

$$\begin{aligned} &|(\vec{\Delta}_{A(\bullet - x)(x)} f)(x)| \\ &\quad = \left| f\left(x + \psi(x) \cdot \frac{A(\bullet - x)(x)}{\psi(x)}\right) - f(x)\right| \\ &\quad \leqslant \sup_{t,\, t + \psi(t) \cdot (A(\bullet - x)(x)/\psi(x)) \in I} \left| f\left(t + \psi(t) \cdot \frac{A(\bullet - x)(x)}{\psi(x)}\right) - f(t)\right| \\ &\quad \leqslant \vec{\omega}_\psi\left(f, \frac{|A(\bullet - x)(x)|}{\psi(x)}\right) \end{aligned}$$

the estimate

$$|(Lf)(x)| \leqslant \vec{\omega}_\psi\left(f, \frac{|A(\bullet - x)(x)|}{\psi(x)}\right) \tag{9}$$

holds true. Finally, inequalities (9) and (8) give the statement of the theorem. ∎

The construction of $\tilde{A} = A + L$ in the proof of Theorem 1 via the operator (3) follows an idea similar to that in [4, proof of Theorem 2.2].

Corollary 1. *Let A, ψ, and ϕ be given as in Theorem 1. If, in addition, ϕ is an admissible step-weight function of the Ditzian–Totik modulus then*

$$|f(x) - (Af)(x)| \leqslant C\left(\omega_\psi\left(f, \frac{|A(\bullet - x)(x)|}{\psi(x)}\right) + \omega_\phi^2\left(f, \frac{\sqrt{A((\bullet - x)^2)(x)}}{\phi(x)}\right)\right) \tag{10}$$

for $x \in I$ and $f \in C(I)$, where the constant C depends only on ψ and ϕ.

Corollary 2. *Let A and ϕ be given as in Theorem 1. If, in addition, ϕ is an admissible step-weight function of the Ditzian–Totik modulus and if A preserves linear functions then*

$$|f(x) - (Af)(x)| \leqslant C\omega_\phi^2\left(f, \frac{\sqrt{A((\bullet - x)^2)(x)}}{\phi(x)}\right)$$

for $x \in I$ and $f \in C(I)$, where the constant C depends only on ϕ.

For a sequence of positive linear operators $A_n\colon C(I) \to C(I)$, $n = 1, 2, \ldots$, Theorem 1 and the corollaries yield the rate of pointwise approximation of functions with respect to the rate of pointwise approximation of x and x^2. Thus the measurement of the error depends on the step-weight functions ψ and ϕ. Because of the relatively free choice of ψ and ϕ we can bridge the gap between the local and the global approximation when in $A_n((\bullet - x)^2)(x)$ the variables n and x can be separated from each other. If we impose the estimation

$$A_n((\bullet - x)^2)(x) \leqslant C_1 \sigma^2(n) \cdot \varphi^2(x) \qquad \text{for} \quad x \in I, \quad n \in N, \tag{11}$$

in which $\varphi\colon I \to R$, φ^2 concave, and $\sigma(n)$ tends to zero for $n \to \infty$ then it follows, by inequality (7) and Corollary 1 with $\psi := \phi$, that

$$|f(x) - (A_n f)(x)| \leqslant C_2\left(\omega_\phi\left(f, \sigma(n)\,\frac{\varphi(x)}{\phi(x)}\right) + \omega_\phi^2\left(f, \sigma(n)\,\frac{\varphi(x)}{\phi(x)}\right)\right). \tag{12}$$

In view of the assumed concavity of φ^2, (12) yields for $\phi=1$ a local and for $\phi=\varphi$ a global estimation of the approximation error. Consequently, (12) bridges both local and global results.

If A_n preserves linear functions, the implication

$$\omega_\phi^2(f,\delta)=O(\delta^\alpha)\ (\delta\to 0)\Rightarrow |f(x)-(A_nf)(x)|\leqslant C\left(\sigma(n)\,\frac{\varphi(x)}{\phi(x)}\right)^\alpha\ (n\to\infty) \tag{13}$$

holds true for $\alpha\in(0,2]$. Unfortunately, if A_n does not preserve linear functions then (13) is fulfilled only for $\alpha\in(0,1)$ for the present because $\omega_\phi^2(f,\delta)=O(\delta^\alpha)$ implies only that $\omega_\phi(f,\delta)=O(\delta^\alpha)$ for $\alpha\in(0,1)$. Thus, the modulus of first order $\omega_\phi(f,\sigma(n)\,\varphi(x)/\phi(x))$ in estimation (12) essentially decelerates the rate of convergence. However, with respect to Theorem 1, this term can also be taken as $\omega_\psi(f,|A(\bullet-x)(x)|/\psi(x))$ for other choices of admissible step-weight functions ψ of the Ditzian–Totik modulus. In the case of Bernstein-type operators it will be seen that the error bound (10) can be improved by suitable choices of ψ.

The question arises as to whether the converse result to (13) also holds true. This question will be discussed in Section 3.

Now we shall illustrate the above theorems by some positive linear operators. Let us first consider the exponential-type operators

$$(L_nf)(x)=\int_I W(n,x,u)\,f(u)\,du,\qquad x\in I,\quad n\in N, \tag{14}$$

where $W(n,x,u)\geqslant 0$ fulfills the conditions

$$\int_I W(n,x,u)\,du=1,\qquad x\in I,\quad n\in N,$$

and

$$\frac{d}{dx}\,W(n,x,u)=\frac{n}{\varphi^2(x)}\,W(n,x,u)(u-x),\qquad x\in I,\quad n\in N,$$

where φ is an analytic function in the interior of I (cf. [12]). L_n is a positive operator since $W(n,x,u)\geqslant 0$. Simple calculations [12] show that L_n preserves linear functions and

$$(L_n(\bullet-x)^2)(x)=\frac{\varphi^2(x)}{n}.$$

In view of the assumed concavity of φ^2, (12) yields for $\phi = 1$ a local and for $\phi = \varphi$ a global estimation of the approximation error. Consequently, (12) bridges both local and global results.

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$$\omega_\phi^2(f, \delta) = O(\delta^\alpha)\ (\delta \to 0) \Rightarrow |f(x) - (A_n f)(x)| \leqslant C\left(\sigma(n)\,\frac{\varphi(x)}{\phi(x)}\right)^\alpha\ (n \to \infty) \tag{13}$$

holds true for $\alpha \in (0, 2]$. Unfortunately, if A_n does not preserve linear functions then (13) is fulfilled only for $\alpha \in (0, 1)$ for the present because $\omega_\phi^2(f, \delta) = O(\delta^\alpha)$ implies only that $\omega_\phi(f, \delta) = O(\delta^\alpha)$ for $\alpha \in (0, 1)$. Thus, the modulus of first order $\omega_\phi(f, \sigma(n)\,\varphi(x)/\phi(x))$ in estimation (12) essentially decelerates the rate of convergence. However, with respect to Theorem 1, this term can also be taken as $\omega_\psi(f, |A(\bullet - x)(x)|/\psi(x))$ for other choices of admissible step-weight functions ψ of the Ditzian–Totik modulus. In the case of Bernstein-type operators it will be seen that the error bound (10) can be improved by suitable choices of ψ.

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and

$$\frac{d}{dx}\, W(n, x, u) = \frac{n}{\varphi^2(x)}\, W(n, x, u)(u - x), \qquad x \in I, \quad n \in N,$$

where φ is an analytic function in the interior of I (cf. [12]). L_n is a positive operator since $W(n, x, u) \geqslant 0$. Simple calculations [12] show that L_n preserves linear functions and

$$(L_n(\bullet - x)^2)(x) = \frac{\varphi^2(x)}{n}.$$

If φ^2 is concave on I then with Corollary 2 the estimate

$$|f(x)-(L_n f)(x)| \leqslant C\omega_\phi^2\left(f, n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right) \tag{15}$$

holds true. In particular this is fulfilled for the operators of Gauss–Weierstrass ($\varphi(x)=1$) Szász-Mirakjan ($\varphi(x)=\sqrt{x}$), and Bernstein ($\varphi(x)=\sqrt{x(1-x)}$). Section 3 shows that an inverse also holds true.

It cannot yet be said whether inequality (12) also holds true for the operators of Baskakov ($\varphi(x)=\sqrt{x(1+x)}$), Post-Widder ($\varphi(x)=x$), and Ismail and May ($\varphi(x)=\sqrt{1+x^2}$). In these cases $\varphi^2(x)$ is not concave.

Let us now consider Bernstein-type operators given as

$$(B_n f)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\,\lambda_{n,k}(f), \qquad x\in[0,1], \tag{16}$$

where $\lambda_{n,k}\in C[0,1]^*$ are bounded positive linear functionals. These operators were considered before in serveral articles (e.g., [11]). For the specific functionals $\lambda_{n,k}(f)=f(k/n)$, $\lambda_{n,k}(f)=(n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt$, and $\lambda_{n,k}(f)=(n+1)\int_0^1 f(t)\,p_{n,k}(t)\,dt$ the operator (16) is the original Bernstein, the Kantorovic, and the Durrmeyer operator, respectively. Obviously, B_n preserves constants if $\lambda_{n,k}(1)=1$ for $k=0, ..., n$. In Section 3, in addition, we will assume that $B_n(\Pi_1)\subset\Pi_1$, where Π_1 is the space of algebraic polynomials of degree at most one, is fulfilled.

It should be pointed out that (11) is not fulfilled for all Bernstein-type operators if we set $\varphi(x)=\sqrt{x(1-x)}$. But the following theorem contains conditions (17) for which the inequality is valid.

THEOREM 2. *Let $\varphi(x):=\sqrt{x(1-x)}$ and ϕ: $[0,1]\to R$ be an admissible step-weight function of the Ditzian–Totik modulus with ϕ^2 concave. Let B_n be defined by (16) and $\lambda_{n,k}\in C[0,1]^*$ be positive linear functionals with $\lambda_{n,k}(1)=1$ for $k=0, ..., n$. If*

$$\lambda_{n,0}(f)=f(0), \qquad \lambda_{n,n}(f)=f(1) \tag{17}$$

and

$$\lambda_{n,k}\left(\left(\bullet-\frac{k}{n}\right)^2\right)\leqslant M\left(\frac{1}{n}\right)^{2\gamma}, \qquad n\in N, \quad k=0, ..., n, \tag{18}$$

for constants $M \geqslant 0$, $\gamma \geqslant 1$ *independent of* n *and* k *then the estimate*

$$|f(x)-(B_n f)(x)| \leqslant C\left(\omega_{\phi^\alpha}\left(f, n^{-\gamma+\alpha/2}\frac{\varphi^\alpha(x)}{\phi^\alpha(x)}\right)+\omega_\phi^2\left(f, n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)\right) \tag{19}$$

holds true for all $x \in [0, 1]$, $f \in C[0, 1]$, *and* $\alpha \in (0, 2]$, *where the constant* C *depends only on* ϕ *and* α.

Proof. Following the same arguments as those in (7), the Cauchy–Schwarz inequality and (18) show that

$$\lambda_{n,k}\left(\bullet-\frac{k}{n}\right) \leqslant \sqrt{\lambda_{n,k}\left(\left(\bullet-\frac{k}{n}\right)^2\right)} \leqslant \frac{\sqrt{M}}{n^\gamma}. \tag{20}$$

By using (17) and (20) we obtain

$$\begin{aligned}
|B_n(\bullet-x)(x)| &= \left|\sum_{k=0}^{n} p_{n,k}(x)\,\lambda_{n,k}(\bullet-x)\right| \\
&= \left|\sum_{k=1}^{n-1} p_{n,k}(x)\,\lambda_{n,k}\left(\bullet-\frac{k}{n}\right)\right| \\
&\leqslant \frac{\sqrt{M}}{n^\gamma}\sum_{k=1}^{n-1} p_{n,k}(x) = \frac{\sqrt{M}}{n^\gamma}(1-x^n-(1-x)^n) \\
&\leqslant \frac{\sqrt{M}}{n^\gamma}(1-x^n-(1-x)^n)^{\alpha/2}
\end{aligned} \tag{21}$$

for $\alpha \in (0, 2]$. If we set $g(x) := 1-x^n-(1-x)^n$ we have

$$g(x) = (1-x)\left(\sum_{k=0}^{n-1} x^k-(1-x)^{n-1}\right) \leqslant n(1-x)$$

for $x \in [0, 1]$. This also implies $g(x) = g(1-x) \leqslant nx$. If we let $x \in [0, \frac{1}{2}]$ then $2(1-x) \geqslant 1$ and $g(x) \leqslant nx \cdot 2(1-x)$. In the same way we obtain $g(x) \leqslant n(1-x) \cdot 2x$ for $x \in [\frac{1}{2}, 1]$ and therefore

$$1-x^n-(1-x)^n \leqslant 2nx(1-x) \qquad \text{for} \quad x \in [0, 1]. \tag{22}$$

From (21) and (22) we obtain

$$|B_n(\bullet-x)(x)| \leqslant 2^{\alpha/2}\sqrt{M}\, n^{-\gamma+\alpha/2}\varphi^\alpha(x), \qquad x \in [0, 1], \quad n \in N. \tag{23}$$

On the other hand, by using the inequality $(r+s)^2 \leqslant 2r^2+2s^2$, $r, s \in R$, we see that

$$\begin{aligned} B_n((\bullet-x)^2)(x) &= \sum_{k=0}^{n} p_{n,k}(x)\,\lambda_{n,k}\left(\bullet-\frac{k}{n}+\frac{k}{n}-x\right)^2 \\ &\leqslant 2\sum_{k=0}^{n} p_{n,k}(x)\,\lambda_{n,k}\left(\bullet-\frac{k}{n}\right)^2+2\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n}-x\right)^2 \\ &\leqslant \frac{2M}{n^{2\gamma}}\sum_{k=1}^{n-1} p_{n,k}(x)+2\,\frac{x(1-x)}{n} \\ &= \frac{2M}{n^{2\gamma}}\,(1-x^n-(1-x)^n)+2\,\frac{x(1-x)}{n}. \end{aligned} \tag{24}$$

With the aid of (22) and assumption $\gamma \geqslant 1$ we obtain

$$B_n((\bullet-x)^2)(x) \leqslant \frac{4M+2}{n}\,\varphi^2(x),$$

which, with (23) and Corollary 1, $\psi := \phi^\alpha$, concludes the proof. ∎

Remark. Especially for $\alpha=1$ Theorem 2 gives

$$|f(x)-(B_n f)(x)| \leqslant C\left(\omega_\phi\left(f, n^{-1/2}\,\frac{\varphi(x)}{\phi(x)}\right)+\omega_\phi^2\left(f, n^{-1/2}\,\frac{\varphi(x)}{\phi(x)}\right)\right). \tag{25}$$

If (18) is fulfilled only for γ with $0 \leqslant \gamma \leqslant 1$ one can obtain (estimating (24) by means of (22)) the weaker estimate

$$|f(x)-(B_n f)(x)| \leqslant C\left(\omega_\phi\left(f, n^{-\gamma+1/2}\,\frac{\varphi(x)}{\phi(x)}\right)+\omega_\phi^2\left(f, n^{-\gamma+1/2}\,\frac{\varphi(x)}{\phi(x)}\right)\right)$$

for $x \in [0, 1]$.

Theorem 2 gives local and global estimations of the approximation error $f(x)-(B_n f)(x)$ if B_n is, with respect to (18), "not far away" from the original Bernstein operator. Condition (17) ensures that $B_n f$ interpolates f at the endpoints 0 and 1.

Estimates of $f(x)-(B_n f)(x)$ in terms of $\Delta_n(x)=\sum_{k=0}^{n} p_{n,k}(x)\,\lambda_{n,k}((\bullet-k/n)^2)$ can be found in [11]. In opposition to our results (via Ditzian–Totik moduli of smoothness with various step-weight functions) the local behaviour of functions in [11] is measured by maximal Lipschitz function $f_\beta^{\sim}(x)$ (see [11, Theorem 2.3]).

Because of the specific choice of the first step-weight function φ^α in Theorem 2 we achieve the following result.

COROLLARY 3. *Let the assumptions of Theorem 2 hold true. If $\phi(x)\leqslant 1$ then the Bernstein-type operators* (16) *fulfill*

$$\omega_\phi^2(f,\delta)\leqslant C_1\delta^\alpha \Rightarrow |f(x)-(B_nf)(x)|\leqslant C_2\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^\alpha \tag{26}$$

for $x\in I$ *and* $\alpha\in(0,\gamma]\backslash\{1\}$, *where the constant* C_2 *depends only on* ϕ, C_1, *and* α.

Proof. Marchaud inequality (see [9, Chap. 4]) gives

$$\omega_\phi(f,\delta)\leqslant C\left\{\delta\int_\delta^c\frac{\omega_\phi^2(f,u)}{u^2}\,du+\|f\|\right\},$$

where $c>0$ is any fixed constant. Consequently, $\omega_\phi^2(f,\delta)=O(\delta^\alpha)$ implies

$$\omega_\phi(f,\delta)=\begin{cases}O(\delta^\alpha), & \alpha\in(0,1)\\ O(\delta\log|\delta|), & \alpha=1\\ O(\delta), & \alpha\in(1,2].\end{cases} \tag{27}$$

If $\alpha\in(0,1)$, then by (25) and (27)

$$|f(x)-(B_nf)(x)|=O\left(\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^\alpha\right)$$

which gives the required statement for $\gamma=1$. To prove the case $\gamma>1$ let $\alpha\in(1,\gamma]$. Assumption $\phi(x)\leqslant 1$ yields $\phi^\alpha(x)\leqslant\phi(x)$, and therefore $\omega_{\phi^\alpha}(f,\delta)\leqslant\tilde{C}\omega_\phi(f,\delta)$ (cf. [9, Chap. 4]). By (27) and Theorem 2 this gives

$$\begin{aligned}|f(x)-(B_nf)(x)|&\leqslant\hat{C}\left\{n^{-\gamma+\alpha/2}\frac{\varphi^\alpha(x)}{\phi^\alpha(x)}+\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^\alpha\right\}\\ &=\hat{C}\left\{n^{-\alpha/2}\frac{\varphi^\alpha(x)}{\phi^\alpha(x)}n^{-\gamma+\alpha}+\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^\alpha\right\}\\ &\leqslant 2\hat{C}\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^\alpha\end{aligned}$$

for $\alpha\in(1,\gamma]$ and $\alpha\leqslant 2$. If $\alpha>2$ then $\omega_\phi^2(f,\delta)=O(\delta^\alpha)$ implies that f is linear (see [9, Chap. 4]). By (23) and $\phi(x)\leqslant 1$ we have

$$|f(x)-(B_nf)(x)|\leqslant\tilde{C}\,|B_n(\bullet-x)|\leqslant\tilde{\tilde{C}}n^{-\alpha/2}\frac{\varphi^\alpha(x)}{\phi^\alpha(x)}$$

for $\alpha\in[2,\gamma]$, which shows that (26) holds true. ∎

We shall now investigate the present situation for Kantorovich-type operators

$$(K_n^I f)(x) = f(0)\, p_{n,0}(x) + \sum_{k=1}^{n-1} p_{n,k}(x)\, \lambda_{n,k}^I(f) + f(1)\, p_{n,n}(x), \qquad x \in [0, 1], \tag{28}$$

whereby the functionals are given by

$$\lambda_{n,k}^I(f) = \frac{1}{I_{n,k}} \int_{I_{n,k}} f(t)\, dt$$

for certain subintervals $I_{n,k} \subset [0, 1]$. For the sake of brevity let us set $[a, b] = [a_{n,k}, b_{n,k}] = I_{n,k}$. Then a short calculation gives

$$\lambda_{n,k}^I\left(\left(\bullet - \frac{k}{n}\right)^2\right) = \frac{1}{3}(b-a)^2 - \left(\frac{k}{n} - a\right)\left(b - \frac{k}{n}\right)$$

and therefore

$$\lambda_{n,k}^I\left(\left(\bullet - \frac{k}{n}\right)^2\right) \leqslant \frac{1}{3}\, |I_{n,k}|^2 \qquad \text{if} \quad \frac{k}{n} \in I_{n,k}.$$

If we assume that $|I_{n,k}| = O(1/n^s)$ for any $s \geqslant 1$, then Corollary 3 shows that

$$\omega_\phi^2(f, \delta) = O(\delta^\alpha) \Rightarrow |f(x) - (K_n^I f)(x)| = O\left(\left(n^{-1/2} \frac{\varphi(x)}{\phi(x)}\right)^\alpha\right) \tag{29}$$

for $\alpha \in (0, s] \backslash \{1\}$. In the case of the original Kantorovich functionals, that is, $I_{n,k} = [k/(n+1), (k+1)/(n+1)]$, we obtain (29) only for $\alpha \in (0, 1)$ since $|I_{n,k}| = O(1/n)$. If we set

$$I_{n,k} = \left[\frac{k}{n} - \frac{1}{n^2}, \frac{k}{n} + \frac{1}{n^2}\right], \qquad k = 1, ..., n-1,$$

then (29) will be valid for $\alpha \in (0, 2] \backslash \{1\}$.

Finally, let us consider Durrmeyer-type operators

$$(D_n^s f)(x) = f(0)\, p_{n,0}(x) + \sum_{k=1}^{n-1} p_{n,k}(x)\, \lambda_{n,k}^s(f) + f(1)\, p_{n,n}(x), \qquad x \in [0, 1], \tag{30}$$

where the functionals are defined by (see [11, 13])

$$\lambda^s_{n,k} f = \frac{\int_0^1 f(t)\, p_{cn,ck}(t)\, dt}{\int_0^1 p_{cn,ck}(t)\, dt}, \qquad c = [n^s], \quad s \geqslant 0.$$

The calculation

$$\begin{aligned}\lambda_{n,k}\left(\left(\bullet - \frac{k}{n}\right)^2\right) &= \lambda_{n,k}(t^2) - 2\frac{k}{n}\lambda_{n,k}(t) + \frac{k^2}{n^2}\\ &= \frac{(ck+1)(ck+2)}{(cn+2)(cn+3)} - 2\frac{k}{n}\frac{ck+1}{cn+2} + \frac{k^2}{n^2}\\ &= \frac{ckn^2 - nck^2 + 2n^2 - 6nk + 6k^2}{(cn+2)(cn+3)\, n^2}\\ &= \frac{k/n - k^2/n^2 + 2/cn - 6k/cn^2 + 6k^2/cn^3}{(1+2/cn)(cn+3)}\end{aligned}$$

establishes

$$\lambda_{n,k}\left(\left(\bullet - \frac{k}{n}\right)^2\right) \leqslant \frac{9}{n^{1+s}}$$

for $n \in N$ and $k = 0, ..., n$. Corollary 3 gives

$$\omega^2_\phi(f, \delta) = O(\delta^\alpha) \Rightarrow |(D^s_n f)(x) - f(x)| = O\left(\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^\alpha\right) \tag{31}$$

for $\alpha \in (0, (1+s)/2] \setminus \{1\}$ and all Durrmeyer-type operators D^s_n with $s \geqslant 1$. It should be noted that the original Durrmeyer functionals are excluded.

3. INVERSE RESULTS

In order to investigate an inverse to (13), a lemma will be needed first. Lemma 1 generalizes a lemma of Becker [1, p. 138] for finite intervals.

LEMMA 1. *Let ϕ: $[a, b] \to R$, $\phi \not\equiv 0$, be a function with ϕ^2 concave. Then for all $x \in [a, b]$, $h > 0$, with $x \pm h \in [a, b]$ the inequality*

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} \frac{ds\, dt}{\phi^2(x+s+t)} \leqslant C\frac{h^2}{\phi^2(x)}$$

holds true, whereby $8 \log 2$ *can be chosen as the constant C.*

Proof. We first show that

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{x+s+t}\leqslant(4\log 2)\cdot\frac{h^2}{x}\qquad\text{for}\quad x\in[h,\infty). \tag{32}$$

If $x\geqslant h$ and $x-h\leqslant h$, we see by means of $(\Delta_h^2 f)(x):=f(x+h)-2f(x)+f(x-h)$ that

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{x+s+t}\leqslant\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{h+s+t}=\Delta_h^2(u\log u)(h)=(2\log 2)\cdot h$$

$$=(4\log 2)\cdot\frac{h^2}{2h}\leqslant(4\log 2)\cdot\frac{h^2}{x}.$$

On the other hand, if $x-h\geqslant h$ then

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{x+s+t}\leqslant\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{x-h}=\frac{2h^2}{x-h+x-h}$$

$$\leqslant\frac{2h^2}{x-h+h}=\frac{2h^2}{x},$$

which proves (32).

Now we need a lower estimate for $\phi^2(x+s+t)$, $x\in(a,b)$. If we let $u\in[a,x]$, then $u=a+\tau(x-a)$, $\tau=(u-a)/(x-a)$, and by concavity

$$\phi^2(u)\geqslant\phi^2(a)+\tau(\phi^2(x)-\phi^2(a))\geqslant\frac{u-a}{x-a}\,\phi^2(x)\qquad\text{for}\quad u\in[a,x]. \tag{33}$$

Likewise, for $u=x+\tau(b-x)\in[x,b]$, $\tau=(u-x)/(b-x)$, we obtain the estimate

$$\phi^2(u)\geqslant\phi^2(x)+\tau(\phi^2(b)-\phi^2(x))\geqslant\frac{b-u}{b-x}\,\phi^2(x)\qquad\text{for}\quad u\in[x,b]. \tag{34}$$

Considering (33) and (34) for $u=x+s+t$ in conjunction with (32) for $x-a, b-x\in[h,\infty)$ we finally obtain

$$\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{\phi^2(x+s+t)}$$

$$=\iint_{\substack{s,\,t\in[-h/2,\,h/2]\\ s+t\leqslant 0}}+\iint_{\substack{s,\,t\in[-h/2,\,h/2]\\ s+t\geqslant 0}}$$

$$\leqslant\frac{x-a}{\phi^2(x)}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{x+s+t-a}+\frac{b-x}{\phi^2(x)}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{b-x-s-t}$$

$$=\frac{x-a}{\phi^2(x)}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{(x-a)+s+t}+\frac{b-x}{\varphi^2(x)}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\frac{ds\,dt}{(b-x)+(s+t)}$$

$$\leqslant(4\log 2)\left(\frac{x-a}{\phi^2(x)}\,\frac{h^2}{x-a}+\frac{b-x}{\phi^2(x)}\,\frac{h^2}{b-x}\right)=(8\log 2)\cdot\frac{h^2}{\phi^2(x)}.\quad\blacksquare$$

THEOREM 3. *Let $I\in\{[0,1], R_0^+, R\}$ and $\phi\colon I\to R$ be an admissible step-weight function of the Ditzian–Totik modulus. Moreover, let $A_n\colon C(I)\to C(I)$, $n\in N$, be bounded positive linear operators so that*

$$|(A_n f)''(x)|\leqslant C_1\frac{n}{\varphi^2(x)}\|f\|\qquad\text{for}\quad x\in I,\quad f\in C(I)\tag{35}$$

and

$$\|\phi^2(A_n g)''\|\leqslant C_2\|\phi^2 g''\|\qquad\text{for}\quad g'\in AC_{loc}(I),\tag{36}$$

where ϕ^2, φ^2, and φ^2/ϕ^2 are concave functions on I. Then for $f\in C(I)$ and $\alpha\in(0,2)$ the pointwise approximation

$$|f(x)-(A_n f)(x)|\leqslant C_3\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^{\alpha},\qquad x\in I,\quad n=1,2,...,\tag{37}$$

implies

$$\omega_\phi^2(f,\delta)\leqslant C_4\delta^\alpha,\qquad\delta>0.$$

Proof. Let $x\in I$ and h so that $x\pm h\in I$ and let $(\Delta_h^2 f)(x)=f(x+h)-2f(x)+f(x-h)$. Both summands in

$$|(\Delta_h^2 f)(x)|\leqslant|(\Delta_h^2(f-A_n f))(x)|+|(\Delta_h^2 A_n f)(x)|$$

shall be estimated. In order to estimate the first summand, we can use the fact that φ^2/ϕ^2 is concave. If we let $u=c+(1/2)(d-c)\in[c,d]$ for any $[c,d]\subset I$ then

$$\frac{\phi^2(u)}{\varphi^2(u)}\geqslant\frac{\phi^2(d)}{2\varphi^2(d)}\qquad\text{and}\qquad\frac{\phi^2(u)}{\varphi^2(u)}\geqslant\frac{\phi^2(c)}{2\varphi^2(c)},$$

showing for $c = x - h$, $d = x + h$, and $u = x$ the inequalities

$$\frac{\varphi^2(x-h)}{\phi^2(x-h)} \leqslant 2\, \frac{\varphi^2(x)}{\phi^2(x)} \qquad \text{and} \qquad \frac{\varphi^2(x+h)}{\phi^2(x+h)} \leqslant 2\, \frac{\varphi^2(x)}{\phi^2(x)}. \tag{38}$$

Thus, in view of assumption (37), we obtain

$$\begin{aligned} |(\Delta_h^2(f - A_n f))(x)| &\leqslant C_3 \left(\left(\frac{\varphi(x+h)}{\phi(x+h)} \right)^{\alpha} + 2 \left(\frac{\varphi(x)}{\phi(x)} \right)^{\alpha} + \left(\frac{\varphi(x-h)}{\phi(x-h)} \right)^{\alpha} \right) n^{-\alpha/2} \\ &\leqslant (2\sqrt{2} + 2)\, C_3 \left(\frac{\varphi(x)}{\phi(x)} \right)^{\alpha} n^{-\alpha/2}. \end{aligned} \tag{39}$$

By reason of the fact that $K_\phi^2(f, \delta^2)$ and $\omega_\phi^2(f, \delta)$ are equivalent we can choose $g = g_\delta \in AC_{loc}(I)$, $\delta \geqslant 0$, so that

$$\|f - g\| \leqslant A\omega_\phi^2(f, \delta) \qquad \text{and} \qquad \|\phi^2 g''\| \leqslant B\, \delta^{-2} \omega_\phi^2(f, \delta).$$

From (35) and (36) for $y \in I$ we see that

$$\begin{aligned} |(A_n f)(y)''| &\leqslant |(A_n(f-g))''\,(y)| + |(A_n g)''\,(y)| \\ &\leqslant \frac{C_1 n}{\varphi^2(y)} \|f - g\| + \frac{C_2}{\phi^2(y)} \|\phi^2 g''\| \\ &\leqslant C_4 \left(\frac{n}{\varphi^2(y)} + \frac{1}{\delta^2 \phi^2(y)} \right) \omega_\phi^2(f, \delta). \end{aligned}$$

Use of Lemma 1 gives

$$\begin{aligned} &|(\Delta_h^2 A_n f)(x)| \\ &\quad = \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} (A_n f)(x + s + t)''\, ds\, dt \\ &\quad \leqslant C_4 \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} \left(\frac{1}{\varphi^2(x+s+t)} + \frac{1}{\delta^2 \phi^2(x+s+t)} \right) ds\, dt\, \omega_\phi^2(f, \delta) \\ &\quad \leqslant C_5 \left(\frac{nh^2}{\varphi^2(x)} + \frac{h^2}{\delta^2 \phi^2(x)} \right) \omega_\phi^2(f, \delta) \end{aligned}$$

for all x with $x \pm h \in I$. Replacing h by $h\phi(x)$ gives

$$|(\Delta^2_{h\phi(x)} A_n f)(x)| \leqslant C_5 \left(\frac{nh^2\phi^2(x)}{\varphi^2(x)} + \frac{h^2}{\delta^2} \right) \omega^2_\phi(f, \delta)$$

for all x with $x \pm h\phi(x) \in I$. Together with (39) we have

$$|(\Delta^2_{h\phi(x)} f)(x)| \leqslant C_6 \left[\left(\frac{\varphi(x)}{\phi(x)} \right)^\alpha n^{-\alpha/2} + \left(\frac{nh^2\phi^2(x)}{\varphi^2(x)} + \frac{h^2}{\delta^2} \right) \omega^2_\phi(f, \delta) \right].$$

Now choose n so that

$$\frac{\varphi(x)}{\phi(x)} n^{-1/2} \leqslant \delta \leqslant 2 \cdot \frac{\varphi(x)}{\phi(x)} n^{-1/2}.$$

Thus

$$|(\Delta^2_{h\phi(x)} f)(x)| \leqslant C_7 \left(\delta^\alpha + \frac{h^2}{\delta^2} \omega^2_\phi(f, \delta) \right)$$

for all x with $x \pm h\phi(x) \in I$. Taking supremum over all h with $0 < h \leqslant t$ we obtain

$$\omega^2_\phi(f, t) \leqslant C_7 \left(\delta^\alpha + \frac{t^2}{\delta^2} \omega^2_\phi(f, \delta) \right), \qquad 0 < t \leqslant \delta,$$

which yields the assertion of the theorem by the well-known Berens and Lorentz lemma [2]. ∎

The assumption of Theorem 3 that "φ^2/ϕ^2 is concave" is needed only for the purpose of deriving the estimations in (38) and (39). Therefore the statement of Theorem 3 also holds true if the concavity of φ^2/ϕ^2 is replaced by the inequality $(\varphi^2(x+t))/(\phi^2(x+t)) \leqslant C(\varphi^2(x)/\phi^2(x))$, for $x, x+t \in I$.

We now investigate the operators considered in Section 1. We begin with the exponential-type operators L_n given in (14) whereby

φ^2 is a polynomial of degree at most 2 without a double zero, $\varphi^2(x) \neq 0$ inside of I and $\varphi(x) = 0$ for finite endpoints of I (40)

and

$$J := \frac{n^2}{\varphi^4(x)} \int_\alpha^\beta \left[\frac{(\varphi^2(x))'}{n} - (u-x) \right] (u-x)^3 \, W(n, x, u) \, du \leqslant M, \qquad (41)$$

where $\alpha = \min\{x, x + (\varphi^2(x))'/n\}$, $\beta = \max\{x, x + (\varphi^2(x))'/n\}$ and M must be an absolute constant independent of n and x. Condition (41) has been imposed by [15].

LEMMA 2. *Then for every concave function $\phi^2 \colon I \to R$ the estimate*

$$\|\phi^2 (L_n g)''\| \leqslant C \, \|\phi^2 g''\|, \qquad g' \in AC_{loc}(I), \qquad n \in N,$$

holds true for any constant C independent of n, g, and ϕ.

Proof. By Taylor's formula we have

$$(L_n g)''(x)$$
$$= \int_I \left[\frac{d^2}{dx^2} W(n, x, u) \right] \left\{ g(x) + g'(x)(u - x) + \int_x^u (u - s)\, g''(s)\, ds \right\} du$$

and by reason of

$$\int_I \left[\frac{d^2}{dx^2} W(n, x, u) \right] (u - x)^i \, du = 0 \qquad \text{for} \quad i = 0, 1,$$

we obtain,

$$(L_n g)''(x) = \int_I \left[\frac{d^2}{dx^2} W(n, x, u) \right] \int_x^u (u - s)\, g''(s)\, ds\, du.$$

By positivity of L_n and (5)

$$\begin{aligned} |\phi(x)^2 (L_n g)''(x)| &\leqslant \|\phi^2 g''\| \int_I \left| \frac{d^2}{dx^2} W(n, x, u) \right| \left| \int_x^u \frac{|u - s|}{\phi^2(s)} \, ds \right| du\, \phi^2(x) \\ &\leqslant \|\phi^2 g''\| \int_I \left| \frac{d^2}{dx^2} W(n, x, u) \right| \left| \int_x^u |u - x| \, ds \right| du \\ &\leqslant \|\phi^2 g''\| \int_I \left| \frac{d^2}{dx^2} W(n, x, u) \right| (u - x)^2 \, du. \end{aligned}$$

Satô showed in [15, Sect. 3] that in view of (41) the integral on the right hand side can estimated by $4 + (\varphi^2(x))''/n + 2J$ (J given by (41)). This gives the required inequality. ∎

THEOREM 4. *Let L_n, $n \in N$, be the exponential-type operators* (14) *satisfying* (40) *and* (41). *If ϕ^2, φ^2, and φ^2/ϕ^2 are concave functions on I then the statements*

(i) $|f(x)-(L_n f)(x)| = O\left(\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^{\alpha}\right), \qquad x \in I, \quad n = 1, 2, ...,$

(ii) $\omega_{\phi}^2(f, \delta) = O(\delta^{\alpha}), \qquad \delta > 0,$

are equivalent for $f \in C(I)$ *and* $\alpha \in (0, 2)$.

Proof. By reason of

$$|(L_n f)''(x)| \leqslant C_1 \frac{n}{\varphi^2(x)} \|f\|$$

(see [15, p. 40]) it follows, with Theorem 3 and Lemma 2, that (i) implies (ii). Implication (ii) $\Rightarrow$ (i) follows immediately from (15). ∎

Theorem 4 combines the local ($\phi = 1$) and the global $\phi = \varphi$) estimates in direct and converse cases. For Bernstein ($\varphi(x) = \sqrt{x(1-x)}$) and Szász-Mirakjan ($\varphi(x) = \sqrt{x}$) operators in particular we then have

$$|f(x)-(L_n f)(x)| = O((n^{-1/2}\varphi^{(1-\lambda)}(x))^{\alpha}) \Leftrightarrow \omega_{\varphi^{\lambda}}^2(f, \delta) = O(\delta^{\alpha}),$$

for all $\lambda \in [0, 1]$ and $\alpha \in (0, 2)$. The question is open as to whether the equivalence holds true for Baskakov, Post-Widder, and Ismail and May operators.

For the Bernstein-type operators given in (16) we show

LEMMA 3. *Let* $\varphi(x) = \sqrt{x(1-x)}$, B_n *be defined by* (16), $\lambda_{n,k} \in C[0, 1]^*$ *positive linear functionals with* $\lambda_{n,k}(1) = 1$ *for* $k = 0, ..., n$ *and* $B_n(\Pi_1) \subset \Pi_1$. *Then for all* $f \in C[0, 1]$

$$|(B_n f)''(x)| \leqslant \frac{2n}{\varphi^2(x)} \|f\|, \qquad x \in [0, 1], \quad n \in N. \tag{42}$$

If

$$\lambda_{n,k}\left(\left(\bullet - \frac{k}{n}\right)^2\right) \leqslant M\left(\frac{1}{n}\right)^2, \qquad n \in N, \quad k = 0, ..., n,$$

for any constant $M \geqslant 0$ *independent of* n *and* k *then*

$$\|\phi^2(B_n g)''\| \leqslant C\, \|\phi^2 g''\|, \qquad n \in N, \tag{43}$$

holds true for $g' \in AC[0, 1]$ *and* ϕ: $[0, 1] \to R$ *with* ϕ^2 *concave. The constant* C *in* (43) *can be replaced by* $12M + 16\sqrt{M} + 8$.

Proof. Let $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$. By differentiating (16) we obtain (following [3, Chapt. 10, Sect. 5])

$$\begin{aligned}|(B_nf)''(x)| &= \left|\left(\frac{1}{\varphi^2(x)}\sum_{k=0}^{n}(k-nx)\,p_{n,k}(x)\,\lambda_{n,k}(f)\right)'\right| \\ &= \left|\frac{1}{\varphi^4(x)}\sum_{k=0}^{n}((k-nx)^2-(1-2x)\,k-nx^2)\,p_{n,k}(x)\,\lambda_{n,k}(f)\right| \\ &\leqslant \frac{\|\lambda_{n,k}\|\,\|f\|}{\varphi^4(x)}\sum_{k=0}^{n}|(k-nx)^2-(1-2x)\,k-nx^2|\,p_{n,k}(x) \\ &\leqslant \frac{\|\lambda_{n,k}\|\,\|f\|}{\varphi^4(x)}\cdot 2n\varphi^2(x),\end{aligned}$$

which shows (since $\|\lambda_{n,k}\|=1$) the first stated inequality (42). The second inequality (43) is obviously fulfilled for $n=1$. Hence we assume $n\geqslant 2$. We use $p'_{n,k}(x)=n(p_{n-1,k-1}(x)-p_{n-1,k}(x))$ to obtain the representation (which has been used also in [11, (2.15)])

$$(B_ng)''(x)=n(n-1)\sum_{k=0}^{n-2}p_{n-2,k}(x)(\lambda_{n,k+2}(g)-2\lambda_{n,k+1}(g)+\lambda_{n,k}(g)). \tag{44}$$

Due to the fact that $\lambda_{n,k+2}(f)-2\lambda_{n,k+1}(f)+\lambda_{n,k}(f)=0$ for $f(x)=1$ and $f(x)=x$, which immediately follows from (44) and $B_n(\Pi_1)\subset\Pi_1$, we derive by Taylor's formula

$$g(t)=g\left(\frac{k+1}{n}\right)+g'\left(\frac{k+1}{n}\right)\left(t-\frac{k+1}{n}\right)+\int_{(k+1)/n}^{t}(t-s)\,g''(s)\,ds$$

for $g'\in AC[0,1]$ that

$$\begin{aligned}(B_ng)''(x)&=n(n-1)\sum_{k=0}^{n-2}p_{n-2,k}(x)\\ &\qquad\times(\lambda_{n,k+2}-2\lambda_{n,k+1}+\lambda_{n,k})\left(\int_{(k+1)/n}^{\bullet}(\bullet-s)\,g''(s)\,ds\right).\end{aligned}$$

Use of the positivity of the functionals $\lambda_{n,j}$ and inequality (5) for concave functions ϕ^2 gives

$$|\phi^2(x)(B_n g)''(x)| \leqslant \phi^2(x) \, \|\phi^2 g''\| \, n(n-1) \sum_{k=0}^{n-2} p_{n-2,k}(x)$$
$$\times (\lambda_{n,k+2} + 2\lambda_{n,k+1} + \lambda_{n,k}) \left(\int_{(k+1)/n}^{t} \frac{|t-s|}{\phi^2(s)} \, ds \right)$$
$$\leqslant \sum_{k=0}^{n-2} \frac{\phi^2(x)}{\phi^2((k+1)/n)} \, \|\phi^2 g''\| \, n(n-1) \, p_{n-2,k}(x)$$
$$\times (\lambda_{n,k+2} + 2\lambda_{n,k+1} + \lambda_{n,k}) \left(t - \frac{k+1}{n} \right)^2. \tag{45}$$

The summand on the right hand side can be estimated as (using (20))

$$\lambda_{n,k+i} \left(t - \frac{k+1}{n} \right)^2 = \lambda_{n,k+i} \left(\left(t - \frac{k+i}{n} \right)^2 - 2 \, \frac{1-i}{n} \left(t - \frac{k+i}{n} \right) + \frac{(1-i)^2}{n^2} \right)$$
$$\leqslant M \frac{1}{n^2} + 2\sqrt{M} \, \frac{|1-i|}{n^2} + \frac{(1-i)^2}{n^2} \qquad \text{for} \quad i = 0, 1, 2$$

and consequently this gives

$$(\lambda_{n,k+2} + 2\lambda_{n,k+1} + \lambda_{n,k}) \left(t - \frac{k+1}{n} \right)^2 \leqslant (3M + 4\sqrt{M} + 2) \frac{1}{n^2}. \tag{46}$$

In order to estimate, $\phi^2(x)/\phi^2((k+1)/n)$ in (45) we show the following two inequalities. Let $a \in [0, 1]$. Then

$$\frac{\phi^2(x)}{\phi^2(a)} \leqslant \frac{1-x}{1-a} \qquad \text{for} \quad x \in [0, a], \tag{47}$$
$$\frac{\phi^2(x)}{\phi^2(a)} \leqslant \frac{x}{a} \qquad \text{for} \quad x \in [a, 1]. \tag{48}$$

Now if we let $x \in [0, a]$, then $a = x + \tau(1-x)$, $\tau = (a-x)/(1-x)$, and

$$\phi^2(a) \geqslant \phi^2(x) + \tau(\phi^2(1) - \phi^2(x)) \geqslant (1-\tau) \, \phi^2(x) = \frac{1-a}{1-x} \, \phi^2(x),$$

showing that (47) holds true. In the same way, inequality (48) is established by

$$\phi^2(a) \geqslant \phi^2(0) + \tau(\phi^2(x) - \phi^2(0)) \geqslant \tau\phi^2(x) = \frac{a}{x} \, \phi^2(x)$$

for $x \in [a, 1]$, $a = 0 + \tau(x - 0)$ with $\tau = a/x$. If $x \in [0, (k+1)/n]$ then (47) with $a = (k+1)/n$ gives

$$\begin{aligned}
&\frac{\phi(x)^2}{\phi^2((k+1)/n)}\, p_{n-2,k}(x) \\
&\quad \leqslant \frac{1-x}{1-(k+1)/n}\, p_{n-2,k}(x) \\
&\quad = \frac{n}{n-1-k}\, \frac{(n-2)}{k!(n-2-k)!}\, x^k(1-x)^{n-1-k} \\
&\quad \leqslant 2\, \frac{(n-1)!}{k!(n-1-k)!}\, x^k(1-x)^{n-1-k} \\
&\quad = 2p_{n-1,k}(x) \qquad \text{for} \quad n \geqslant 2. \qquad (49)
\end{aligned}$$

If $x \in [(k+1)/n, 1]$ we deduce from (48), $a = (k+1)/n$, that

$$\begin{aligned}
&\frac{\phi(x)^2}{\phi^2((k+1)/n)}\, p_{n-2,k}(x) \\
&\quad \leqslant \frac{nx}{k+1}\, p_{n-2,k}(x) \\
&\quad = \frac{n}{k+1}\, \frac{(n-2)!}{k!(n-2-k)!}\, x^{k+1}(1-x)^{n-2-k} \\
&\quad \leqslant 2\, \frac{(n-1)!}{(k+1)!(n-2-k)!}\, x^{k+1}(1-x)^{n-2-k} \\
&\quad = 2p_{n-1,k+1}(x) \qquad \text{for} \quad n \geqslant 2. \qquad (50)
\end{aligned}$$

Now taking (45), (46), (49), and (50) into account, we obtain

$$\begin{aligned}
|\phi^2(x)(B_n g)''(x)| &\leqslant \|\phi^2 g''\|\, \frac{n(n-1)}{n^2} \sum_{k=0}^{n-2} 2(p_{n-1,k}(x) + p_{n-1,k+1}(x)) \\
&\quad \times (3M + 4\sqrt{M} + 2) \\
&\leqslant \|\phi^2 g''\|\, 4(3M + 4\sqrt{M} + 2),
\end{aligned}$$

which completes the proof of the lemma. ▮

Finally, we arrive at

THEOREM 5. *Let* $\varphi(x) = \sqrt{x(1-x)}$ *and* $\phi\colon [0, 1] \to [0, 1]$ *be an admissible step-weight function of the Ditzian–Totik modulus.*

Let B_n *be the Bernstein-type operators* (16) *with positive linear functionals* $\lambda_{n,k} \in C[0,1]^*$ *satisfying* $\lambda_{n,0}(f) = f(0)$, $\lambda_{n,n}(f) = f(1)$, $\lambda_{n,k}(1) = 1$, $k = 0, ..., n$, *and*

$$\lambda_{n,k}\left(\left(\bullet - \frac{k}{n}\right)^2\right) \leqslant M\left(\frac{1}{n}\right)^{2\gamma}, \qquad n \in N, \quad k = 0, ..., n,$$

for constants $M \geqslant 0$, $1 \leqslant \gamma \leqslant 2$ *independent of* n *and* k. *Moreover, let* $B_n(\Pi_1) \subset \Pi_1$. *If* ϕ^2 *and* φ^2/ϕ^2 *are concave functions on* $[0,1]$ *then*

$$|f(x) - (B_n f)(x)| = O\left(\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^{\alpha}\right), \qquad x \in [0,1], \quad n = 1, 2, ...,$$

and

$$\omega_\phi^2(f, \delta) = O(\delta^\alpha), \qquad \delta > 0,$$

are equivalent for $f \in C[0,1]$ *and* $\alpha \in (0, \gamma)\backslash\{1\}$.

Proof. Corollary 3 and Theorem 3 in combination with Lemma 3 give the equivalence. ∎

Thus the above equivalence is valid for Kantorovich-type operators K_n^I in (28), i.e.,

$$|f(x) - (K_n^I f)(x)| = O\left(\left(n^{-1/2}\frac{\varphi(x)}{\phi(x)}\right)^{\alpha}\right) \Leftrightarrow \omega_\phi^2(f, \delta) = O(\delta^\alpha)$$

for $\alpha \in (0, \gamma)\backslash\{1\}$ if $|I_{n,k}| = O(1/n^\gamma)$, $\gamma \in [1,2]$, and $I_{n,k} = [a_k, b_k]$ are symmetric intervals around the points k/n. Obviously (because of the definition of the functionals) $\lambda_{n,k}(1) = 1$ holds true and because of $\lambda_{n,k}(f) = (a_k + b_k)/2 = k/n$ for $f(x) = x$ the assumption $B_n(\Pi_1) \subset \Pi_1$ of Theorem 5 is valid. The same holds true for Durrmeyer-type operators $D_n^{2\gamma-1}$ if $\gamma \in [1,2]$ (see (30), (31); for $D_n^{2\gamma-1}(\Pi_1) \subset \Pi_1$ see [13]).

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